ICTP PWF - Operators on Hilbert Spaces
Tutorial 2
Discussion on Problem Set 1

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Resources:

1. *Introductory Functional Analysis with Applications*, by Erwin Kreyszig. PDF: https://physics.bme.hu/sites/physics.bme.hu/files/users/BMETE15AF53_kov/Kreyszig%20-%20Introductory%20Functional%20Analysis%20with%20Applications%20(1).pdf
2. Dr Frederic P. Schuller's Lectures on Quantum Theory. Lecture notes: https://tales.mbivert.com/Lectures_on_Quantum_Theory_complete.pdf , YouTube playlist: https://www.youtube.com/playlist?list=PLPH7f_7ZlzxQVx5jRjbfRGEzWY_upS5K6
3. *Real Analysis* (4th Edition), by HL Royden and PM Fitzpatrick. PDF: https://uas201142004vilesia.wordpress.com/wp-content/uploads/2014/12/bahan-ajar-analisis-real-3.pdf

I.1. Let $\mathbb{R}$ be the set of real numbers.

(a) What is a dense subset of $\mathbb{R}$?

(b) Is the set of irrational numbers dense in $\mathbb{R}$? If so, give a proof.

I.2. What is a continuous function from $\mathbb{R}$ to $\mathbb{R}$? Give two examples and two non-examples of such functions with explanations.

I.3. Let $C^\infty(\mathbb{R},\mathbb{R})$ be the set of all real-valued smooth functions on $\mathbb{R}$. Is it an inner product space and/or a Hilbert space? Provide arguments in favour of your answer.

I.4. Define the space $L^2(\mathbb{R},\mathbb{C})$ of square-integrable complex-valued functions on $\mathbb{R}$ with respect to the standard measure. Prove that $L^2(\mathbb{R},\mathbb{C})$ is a separable Hilbert space.

I.5. Let $\mathbb{S}$ be the unit circle and let $f \in C^\infty(\mathbb{S},\mathbb{R})$. Define the multiplication operator M_f by f on $L^2(\mathbb{S},\mathbb{R})$. Is it a bounded operator? If so, give a proof.

I.1. Let $\mathbb{R}$ be the set of real numbers.

(a) What is a dense subset of $\mathbb{R}$?

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"$\mathbb{Q}$ is dense in $\mathbb{R}$"

$\overline{\mathbb{Q}} = \mathbb{R}$

$\mathbb{Q} \cup$ all limit points

$\overline{\mathbb{Q}} = \boxed{\mathbb{Q}} \cup \underbrace{\{\text{all limit points of } \mathbb{Q}\}}_{\text{limit of all Cauchy sequences in } \mathbb{Q}} = \mathbb{R}.$

$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, \ldots \longrightarrow \pi \notin \mathbb{Q}$

rational sequence

"$\mathbb{Q}$ is a dense subset of $\mathbb{R}$."

$\mathbb{Q} \cup$ all limit points $= \mathbb{R}$.

X is a space

$A \subseteq X$.

$$\text{Closure of } A = \bar{A} = A \cup \underbrace{\{\text{all limit points of } A\}}_{\text{limit of sequences in } A}.$$

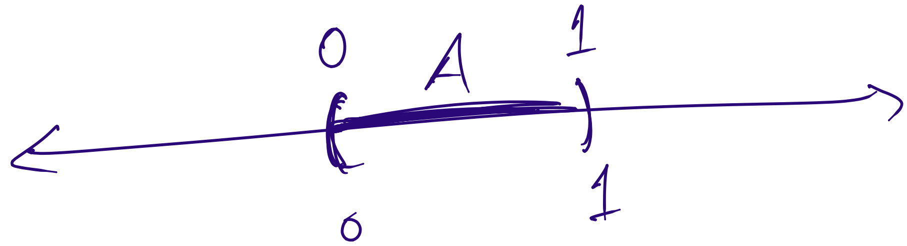
$$\bar{Q} = Q \cup \underbrace{\{\text{limit of sequences in } Q\}}$$

$1, 1.5, 1.5, 1.5, 1.5, \ldots \to 1.5$

$1, -1, 1, -1, 1, -1, \ldots$ NOT convergent

$2, 2.7, 2.71, 2.718, 2.7182, \ldots \to e \notin Q$

$A = (0,1) \subseteq X = \mathbb{R}$



$$\bar{A} = A \cup \{\text{limit points of } A\} = [0,1].$$

0, 1

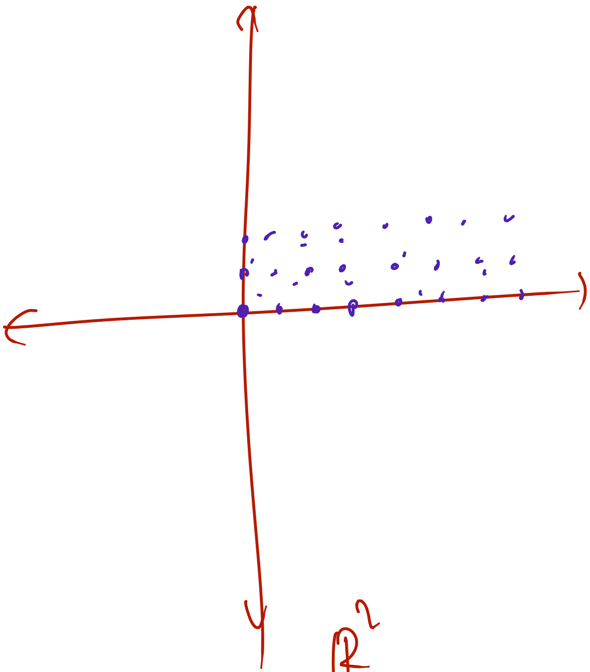
Can you construct a seq a_n s.t. $a_n \in A\ \forall n$; and $a_n \longrightarrow 0$?

$0.1,\ 0.01,\ 0.0001,\ 0.00001,\ 0.000001, \ldots$

$\in (0,1) \longrightarrow 0$

$0.9,\ 0.99,\ 0.999, \ldots \longrightarrow 1$

$\mathbb{Q} \cup \{\text{all limit points}\}$ $\mathbb{R}$



$\mathbb{R}^2$

A X

$\mathbb{Q}^2 \subset \mathbb{R}^2 \longrightarrow (x,y)\ .\ x,y \in \mathbb{R}$

$(a,b)\ ,\ a,b \in \mathbb{Q}.$

$\bar{A} = A \cup \{\text{all limit points of } A\}.$

Claim: $\overline{\mathbb{Q}^2} = \mathbb{R}^2$.

Proof: $\overline{\mathbb{Q}^2} = \mathbb{Q}^2 \cup \{\text{all limit points}\}$

I have to show that all $\boxed{(x,y) \in \mathbb{R}^2}$ belong to $\mathbb{Q}^2 \cup \{\text{all limit points}\}$.

- if both $x,y \in \mathbb{Q}$, then $(x,y) \in \mathbb{Q}^2$
- $x \in \mathbb{R} = \bar{\mathbb{Q}}$, $y \in \mathbb{R} = \bar{\mathbb{Q}}$ } there is a rational seq that converges to x. . . . . y.

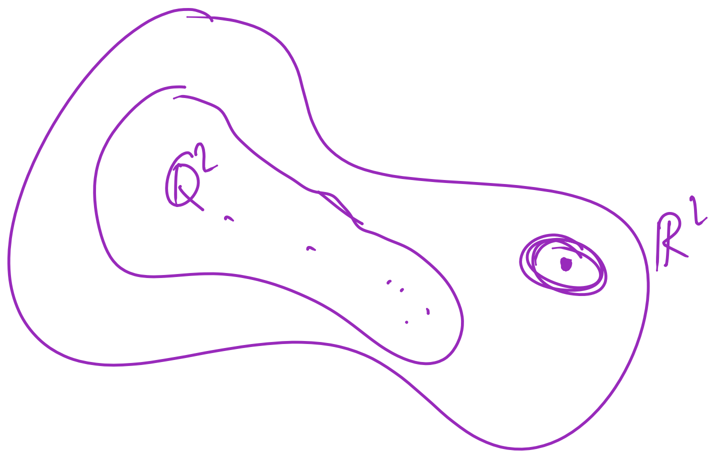
$a_1, a_2, a_3 \ldots \ldots \longrightarrow x$

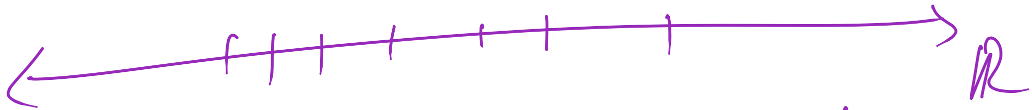
$b_1, b_2, b_3 \ldots \ldots \longrightarrow y$.

Form a seq of pairs: (seq in $\mathbb{Q}^2$)

$(a_1, b_1), (a_2, b_2), (a_3, b_3) \ldots \longrightarrow (x,y) \in \mathbb{R}^2$

Every element of $\mathbb{R}^2$ is a limit of a seq in $\mathbb{Q}^2$.





π approximate using element of $\mathbb{Q}$.

$\mathbb{R} = \overline{\mathbb{Q}}$.

$\mathbb{I} = \{$all irrational numbers$\}$

is it dense in $\mathbb{R}$? $(\overline{\mathbb{I}} = \mathbb{R}?)$

given ANY $x \in \mathbb{R}$ we can find a sequence in $\mathbb{I}$ that converges to x.

How?

$\boxed{(x-1, x+1)}$ a_1 = an irrational number in $(x-1, x+1)$

$(x-\frac{1}{2}, x+\frac{1}{2})$ a_2 = an irrational number in $(x-\frac{1}{2}, x+\frac{1}{2})$

$\vdots$

a_n = an irrational number in $(x-\frac{1}{n}, x+\frac{1}{n})$.

$\downarrow$

x.

Therefore, $\overline{\mathbb{I}} = \mathbb{R}$.

$\boxed{3}$

$\mathbb{Q}$, $\mathbb{I}$ both are dense in $\mathbb{R}$.

$\mathbb{Q} \downarrow$ $\boxed{\text{countable}}$

$\mathbb{I} \downarrow$ uncountable.

Cantor's diagonalization argument.

$$
\begin{array}{cccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \cdots \\
0/2 & 1/2 & 2/2 & 3/2 & 4/2 & 5/2 & \cdot & - & - & & & \\
0/3 & 1/3 & 2/3 & 3/3 & 4/3 & 5/3 & \cdot & - & - & \cdot & &
\end{array}
$$

$0, 1, 0/2, 2, 1/2, 0/3, 3, 2/2, 1/3, \ldots$

$\boxed{\mathbb{Q} \text{ is countable.}}$

Q: If a subset is dense in $\boxed{\mathbb{R}}$, shouldn't it be uncountable in all cases?

NOT necessarily.

Hilbert space: complete inner-prod space.

Separable Hilbert space:

$\boxed{\text{A Hil. Space that has a countable dense subset.}}$

Conclusion: $\mathbb{R}$ is a sep'ble Hilbert space.

Although $\mathbb{Q}$ is countable,

consider $\{\text{all rational sequences}\} = \underline{\mathbb{Q}^{\mathbb{N}}}$

uncountable.

I.2. What is a continuous function from $\mathbb{R}$ to $\mathbb{R}$? Give two examples and two non-examples of such functions with explanations.

any polynomial,
sin, cos, exp, ...

Non-continuous functions:

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

$$f(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q}. \end{cases}$$

I.3. Let $\boxed{C^\infty(\mathbb{R}, \mathbb{R})}$ be the set of all real-valued smooth functions on $\mathbb{R}$. Is it an inner product space and/or a Hilbert space? Provide arguments in favour of your answer.

$$C^\infty(\mathbb{R},\mathbb{R}) = \left\{ \text{all functions } f: \mathbb{R} \to \mathbb{R}, \text{ s.t. } f \text{ is infinitely differentiable} \right\}.$$

Can we put an inn. prod on $C^\infty(\mathbb{R},\mathbb{R})$?

1. $\langle x,y\rangle = \langle y,x\rangle^*$ if we are in a $\mathbb{C}$-Hilbert space,

$\langle x,y\rangle = \langle y,x\rangle$ " " " " $\mathbb{R}$-Hilbert space.

2. $\langle ax+by, z\rangle = a\cdot\langle x,z\rangle + b\langle y,z\rangle$

3. if $x \neq 0$, then $\langle x,x\rangle > 0$. ✓

$$\boxed{\|x\| = \sqrt{\langle x,x\rangle}.}$$

$$\langle f,g\rangle = \cancel{\int_{-\infty}^{\infty} \overline{f(x)}\cdot g(x)\, dx}.$$

$$\boxed{\langle f,g\rangle = \int_{-\infty}^{\infty} f(x)\cdot g(x)\, dx.}$$ ✓

What's wrong!

$f(x) = \sin x$, $g(x) = e^x = \exp(x)$.

$\langle f,g\rangle = \int_{-\infty}^{\infty} (\sin x \cdot e^x)\, dx$ doesn't exist.

x^n

"MANUALLY" COMPUTED ANTIDERIVATIVE:
$\int \mathbf{f}(x)\,\mathbf{d}x = \mathbf{F}^{\star}(x) =$

"Manual" integration with steps:
The calculator finds an antiderivative in a comprehensible way. Note that due to some simplifications, it might only be valid for parts of the function.

$$\frac{e^{x}\left(\sin(x)-\cos(x)\right)}{2}+C$$

does not converge

Show steps

e^{x}

$\lim_{x\to\infty} \sin(x)$ does not exist.

ANTIDERIVATIVE COMPUTED BY MAXIMA:
$\int \mathbf{f}(x)\,\mathbf{d}x = \mathbf{F}(x) =$

$$\frac{e^{x}\left(\sin(x)-\cos(x)\right)}{2}+C$$

Simplify

$\therefore C^{\infty}(\mathbb{R},\mathbb{R})$ can't be given an inner prod.

I.4. Define the space $\boxed{L^2(\mathbb{R},\mathbb{C})}$ of square-integrable complex-valued functions on $\mathbb{R}$ with respect to the standard measure. Prove that $L^2(\mathbb{R},\mathbb{C})$ is a separable Hilbert space.

$f:\mathbb{R}\to\mathbb{C}$

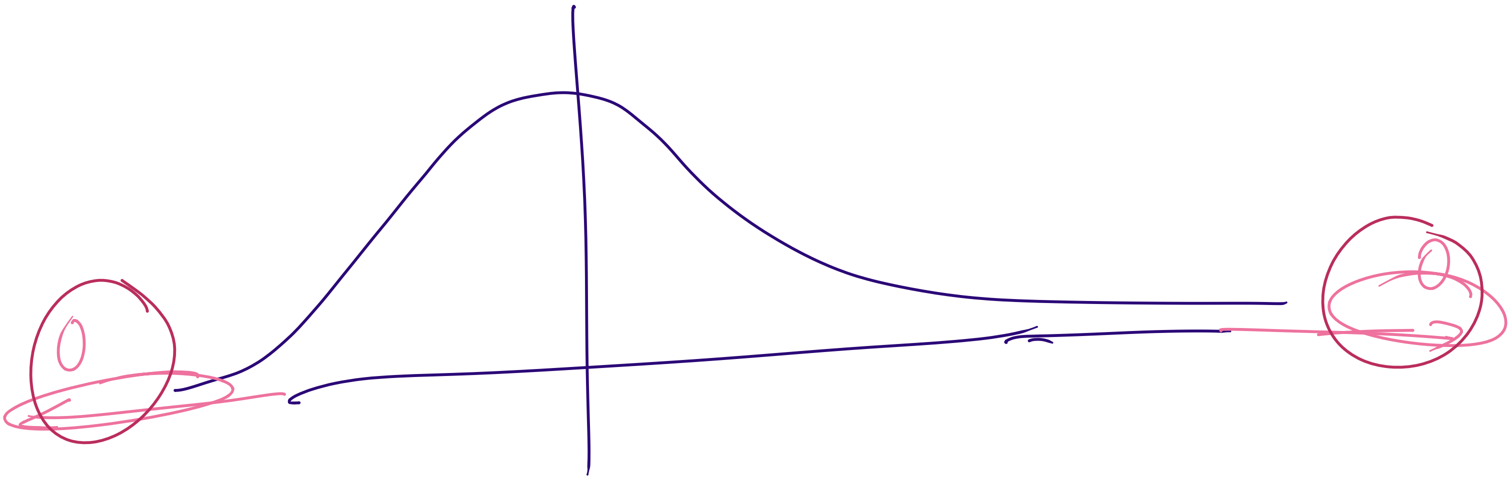
$\int_{-\infty}^{\infty} |f(x)|^2\,dx$ is finite.

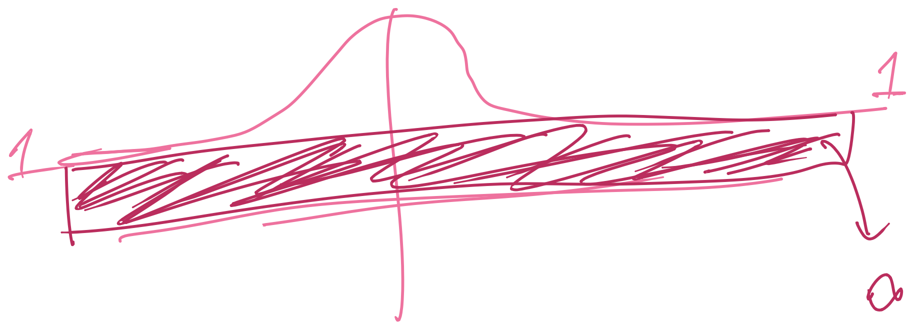
$f(x)=\left(e^{-x^2}\right)$.

$\int_{-\infty}^{\infty} |f(x)|^2\,dx$

$= \int_{-\infty}^{\infty} e^{-2x^2}\,dx = \sqrt{\pi/2} < \infty$.

$\in \boxed{L^2(\mathbb{R},\mathbb{C})}$.





$f(x) = \sin x \notin L^2(\mathbb{R}, \mathbb{C}).$

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} \sin^2 x \, dx$$

doesnt exist

$f(x) = e^{ix} \notin L^2(\mathbb{R}, \mathbb{C})$

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} 1 \, dx = \infty$$

$f(x) = e^{-|x|} \sin x$

Inner product on $L^2(\mathbb{R},\mathbb{C})$

$f, g \in L^2(\mathbb{R},\mathbb{C})$

$$\langle f,g\rangle = \int_{-\infty}^{\infty} \overline{f(x)}\, g(x)\, dx.$$

How do you know this integral exists? or it's a finite number.

Cauchy-Schwarz inequality

$$\boxed{\left(\sum a_i b_i\right)^2 \leq \sum a_i^2 \cdot \sum b_i^2}$$

Integral version:

$$\boxed{\left|\int_{-\infty}^{\infty} f(x)\cdot g(x)\, dx\right|^2 \leq \left[\int_{-\infty}^{\infty} |f(x)|^2\, dx\right] \cdot \left[\int_{-\infty}^{\infty} |g(x)|^2\, dx\right].}$$

$$\boxed{\left|\int_{-\infty}^{\infty} \overline{f(x)}\, g(x)\, dx\right|^2 \leq \underbrace{\int_{-\infty}^{\infty} |f(x)|^2\, dx}_{\text{finite}} \cdot \underbrace{\int_{-\infty}^{\infty} |g(x)|^2\, dx}_{\text{finite}}.}$$

Since $f, g \in L^2(\mathbb{R},\mathbb{C})$, the integrals on RHS are finite. Therefore the LHS is also finite.

$$\langle f, g\rangle = \overline{\langle g, f\rangle}$$

$$= \int_{-\infty}^{\infty} \overline{f(x)}\, g(x)\, dx = \overline{\int_{-\infty}^{\infty} \overline{\overline{f(x)}\, g(x)}\, dx} = \overline{\int_{-\infty}^{\infty} \overline{g(x)}\, f(x)\, dx}$$

$$= \overline{\langle g, f\rangle}.$$

What does $\underbrace{\text{sep'ble Hil space}}$ mean?

$\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$.

there's a countable dense subset.

Other ways of showing it:

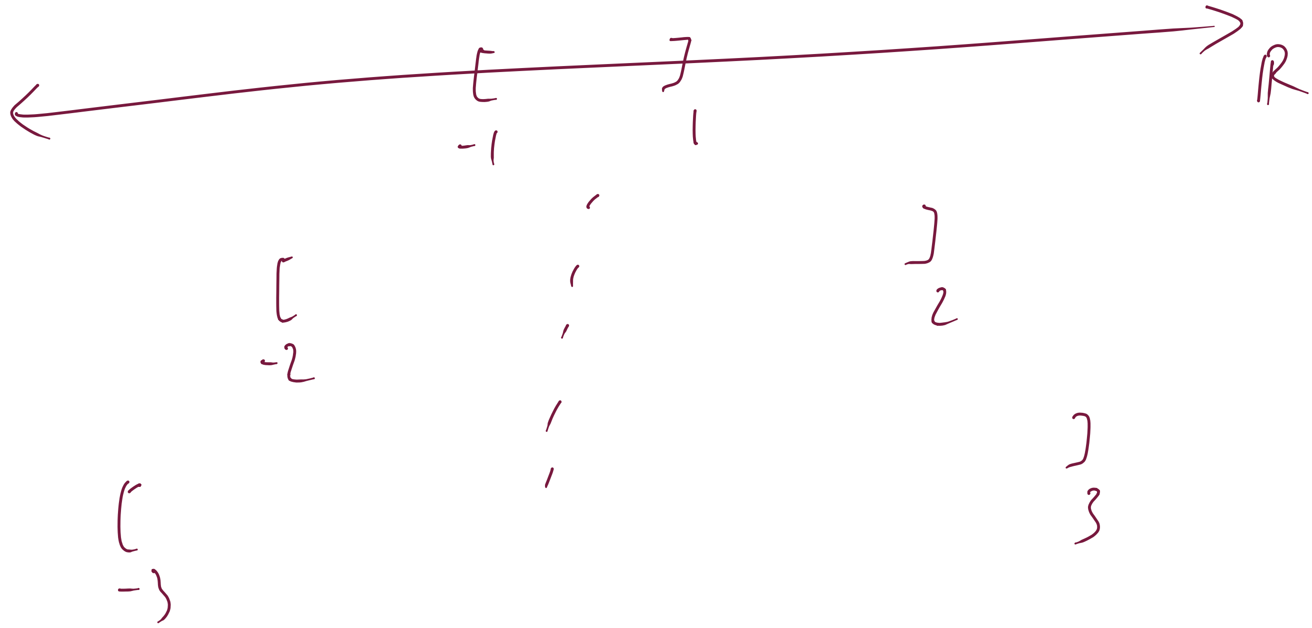
1. Stone-Weierstrass theorem.

   Any continuous function on a closed interval can be approximated using polynomials.

2. Using step functions.

$\boxed{\text{MathStackexchange answer by David Ullrich.}}$

Have to show that $L^2(\mathbb{R}, \mathbb{C})$ has a countable dense subset.



$\mathbb{R}$ can be covered using intervals of the form $[-n, n]$, for $n \in \mathbb{Z}_{>0}$.

Consider a sq. integrable function, f.

$\int_{-\infty}^{\infty} |f(x)|^2 \, dx$ is finite.

$\int_{-n}^{n} |f(x)|^2 \, dx$ is also finite.

$\Rightarrow \boxed{f \in L^2([-n,n], \mathbb{C})}$.

Focus $L^2([-n,n],\mathbb{C})$

$$\Big\downarrow \cong$$

$$L^2([-\pi,\pi],\mathbb{C})$$

$[\,-n \qquad\qquad n\,]$

shrink

$f'(x) = f\left(x\frac{\pi}{n}\right)$

$[\,-\pi \qquad\qquad \pi\,]$

$\boxed{L^2([-\pi,\pi],\mathbb{C})}$ construct an $\boxed{\text{orthonormal basis}}$

A countable orthonormal set such that every element is an infinite linear combination of the orthonormal set.

$\{f_1, f_2, f_3, f_4, \ldots\ldots\}$ orthonormal basis

s.t. any $f \in L^2([-\pi,\pi],\mathbb{C})$

can be written as $\boxed{f = \sum_{n=1}^{\infty} a_n f_n}$

Theorem: A Hil space $\mathcal{H}$ is sep'ble

$\iff$

it contains a countable $\boxed{\text{orthormal basis}}$

pf: (See Kreyszig's book or Schuller's lecture)

What's an orthonormal basis of

$L^2([-\pi,\pi], \mathbb{C})$?

$\left\{ \frac{1}{\sqrt{2\pi}} e^{inx} : n \in \mathbb{Z} \right\}$ an orthonormal ~~set~~ basis.

$$\int_{-\pi}^{\pi} |e^{inx}|^2 = 2\pi$$

$$\int_{-\pi}^{\pi} e^{-inx} e^{imx} \, dx = 0 \quad \text{if } m \neq n.$$

Any $f \in L^2([-\pi,\pi], \mathbb{C})$ can be written

as $$f(x) = \sum_{n=-\infty}^{\infty} \boxed{a_n} e^{inx}$$

I.5. Let $\mathbb{S}$ be the unit circle and let $f \in C^\infty(\mathbb{S},\mathbb{R})$. Define the multiplication operator M_f by f on $L^2(\mathbb{S},\mathbb{R})$. Is it a bounded operator? If so, give a proof.

Yes

$$L^2(S,\mathbb{R}) = \left\{ g : S \to \mathbb{R} \text{ s.t. } \int_S |g|^2 \text{ is finite} \right\}.$$

Another function $f \in C^\infty(S,\mathbb{R})$.

What is M_f?

Operator

$$\boxed{\begin{aligned} M_f : L^2(S,\mathbb{R}) &\longrightarrow L^2(S,\mathbb{R}). \\ g &\longmapsto fg. \end{aligned}}$$

$T : \mathcal{H} \to \mathcal{H}$.

$\|T\| = ?$

$\max \|Tx\|$, where $\|x\| = 1$.

$$\|M_f\|^2 = \max_{\|g\|=1} \|M_f g\|^2$$

$$= \max_{\|g\|=1} \|fg\|^2$$

$$= \max_{\|g\|=1} \int_{S'} |fg|^2 \, d\theta$$

$L^2(S,\mathbb{R})$

$\|g\|^2 = \int |g|^2$

$$\leq \max_{\|g\|=1} \int_{S^1} M^2 |g|^2 \, d\theta .$$

$$\leq M^2 \max_{\|g\|=1} \int_{S^1} |g|^2 = 1$$

$f : S^1 \to \mathbb{R}$
S^1 is compact
f has a max & a min value

$|f|$ has a max, say M.

$$\Rightarrow \boxed{\|M_f\| \leq M.}$$

is a bounded operator.