Operators on Hilbert Spaces

REF

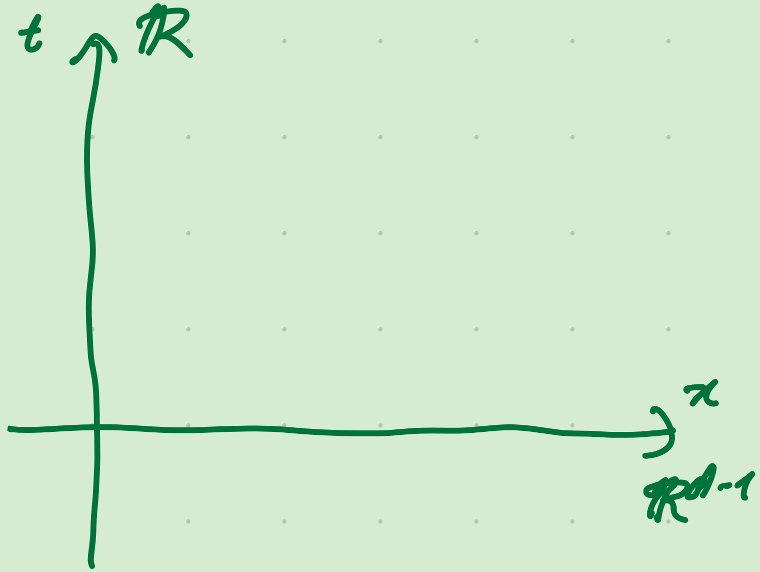
- M. REED & B. SIMON, METHODS OF MODERN MATHEMATICAL PHYSICS - I
- K. SCHMÜDGEN, UNBOUNDED SA OP ON HILBERT SPACE
- D. BORTHWICK, SPECTRAL THEORY

MOTIVATION

WAVE EQ $\quad \left(\frac{\partial^2}{\partial t^2} - \Delta\right) u = 0, \quad u \in C^\infty(\mathbb{R}^d, \mathbb{C})$ ($\mathbb{R}$), $\quad u(0,x) = f$, $\quad \frac{\partial u}{\partial t}(0,x) = g$

$$\Rightarrow u(t,x) = \cos\left(t\sqrt{-\Delta}\right) f(x) + \sqrt{-\Delta}^{-1} \sin\left(t\sqrt{\Delta}\right) g(x)$$

loc



HEAT EQ $\quad \left(\frac{\partial}{\partial t} - \Delta\right) u = 0, \quad u(0,x) = f$

$$\Rightarrow u(t,x) = e^{t\Delta} f(x)$$

$C^k(\mathbb{R}^d, \mathbb{K}), \ \mathbb{K} := \mathbb{R}, \mathbb{C}$

SCHRÖDINGER EQ $\left(i\frac{\partial}{\partial t} - H\right) u = 0$, $u(0,x) = f$

$$\Rightarrow u(t,x) = e^{-itH} f(x)$$

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

new def?

$$f : A \to \overset{A}{B} \to \qquad g : B \to C$$

$$f^2 \cancel{:=} f \circ f$$

$$A \xrightarrow{f} B \xrightarrow{g} C$$

$$\Delta = \frac{\partial^2}{\partial x^2} : C^2(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$$

$\cup X$

Dom Δ

x^2

$$\frac{dx^2}{dx} = 2x$$

$$\frac{d^2x^2}{dx^2} = 2$$

WHAT IS $f(T)$?

CANONICAL COMMUTATION RELATIONS (CCRs) $[\hat{x}, \hat{p}] = i\hbar \mathbb{1}$

$A \qquad A_t$

$f : A \ni \xrightarrow{\text{SET}} \in B, \quad x \longmapsto y = f(x)$ $\qquad :=$

$\Rightarrow$

$A \longmapsto f(A) := ?$

$PDO^m(\mathbb{R}^d) := \{$ SPACE OF PDO OF ORDER m ON $\mathbb{R}^d \}$

$\Delta \in PDO^2(\mathbb{R}^d), \qquad \Delta : C^2(\mathbb{R}^d) \longrightarrow$

$\Box \in PDO^2(\mathbb{R}^d) \qquad \Box : C^2(\mathbb{R}^d) \longrightarrow$

$H \in PDO^1(\mathbb{R}^d) \qquad H : C^1(\mathbb{R}^d) \longrightarrow$

$\mathbb{C}^d$

$z, \ \bar{\cdot} : z \mapsto \bar{z}$

$|z| := \sqrt{\bar{z} \cdot z}$

$(V, \mathbb{K} = \mathbb{C})$

NORMED VECTOR SPACE

$(V, \mathbb{C})$ ... COMPLEX VECTOR SPACE

$\| \cdot \| : V \longrightarrow \mathbb{R}$

$\forall u, v \in V, \quad c \in \mathbb{C}:$

(I) $\|v\| \geq 0, \quad \|v\| = 0 \Leftrightarrow v = 0$

(II) $\|cv\| = |c| \, \|v\|, \quad \|u + v\| \leq \|u\| + \|v\|$

· <u>EXM</u> $K \subset \mathbb{R}^d$ CPT

$C(K,\mathbb{R}) := \{ f : K \to \mathbb{R} \text{ CONTINUOUS} \}$

$\|\cdot\| : C(K,\mathbb{R}) \to \mathbb{R}, \quad f \mapsto \|f\| := \sup_{x \in K} |f(x)|$

$(C(K,\mathbb{R}), \|\cdot\|)$

$(V, \|\cdot\|)$

$\operatorname{dist}(u,v) := \|u - v\|$

$(u_n)_n \in V$ CONVERGES TO u IF

$\lim_{n \to \infty} \|u_n - u\| = 0$

$(v_n)_n \in V$ ... CAUCHY IF

$\lim_{n,m \to \infty} \|v_n - v_m\| = 0$

? DOES EVERY CAUCHY SEQUENCE IN V CONVERGE IN V ?

$\longrightarrow$ NO

IF YES $\Longrightarrow$ $(V, \|\cdot\|)$ ... COMPLETE AS A METRIC SPACE

· <u>DEF</u> BANACH SPACE := A COMPLETE NORMED VS

INNER PRODUCT SPACE

· <u>DEF</u> $(V, \mathbb{K})$

AN INNER PRODUCT ON V

$$\langle \cdot | \cdot \rangle : V \times V \longrightarrow \mathbb{K}, \quad (u, v) \longmapsto \langle u | v \rangle$$

$$\langle c\, u + v, w \rangle = \bar{c} \langle u | w \rangle + \langle v | w \rangle$$

$$\langle u | v \rangle = \overline{\langle v | u \rangle}$$

$$\langle v|v\rangle \geq 0$$

$$\langle v|v\rangle = 0 \iff v = 0$$

$(V, \langle\cdot|\cdot\rangle)$ ... INNER PRODUCT SPACE

· PROPERTIES

· CAUCHY-SCHWARTZ INEQUALITY

$$|\langle u|v\rangle| \leq \sqrt{\langle u|u\rangle}\,\sqrt{\langle v|v\rangle}$$

· $\|v\| := \sqrt{\langle v|v\rangle}$

· $$\langle u|v\rangle \overset{\mathbb{K}=\mathbb{C}}{=} \frac{1}{4}\left(\underbrace{\|u+v\|^2 - \|u-v\|^2} + i\,\|u+iv\|^2 - i\,\|u-iv\|^2\right)$$

$\mathbb{K} = \mathbb{R}$

· **DEF** HILBERT SPACE := COMPLETE INNER PRODUCT SPACE
$(\mathcal{H}, \langle \cdot | \cdot \rangle)$

$(V, \|\cdot\|, \widetilde{\|\cdot\|})$

$\|\cdot\|$ & $\widetilde{\|\cdot\|}$ ... EQUIVALENT IF

$\exists\ cst, \widetilde{cst} > 0 \mid cst\, \|\cdot\| \leqslant \widetilde{\|\cdot\|} \leqslant \widetilde{cst}\, \|\cdot\|$

dist, $\widetilde{\text{dist}}$

· **THM** $(V, \|\cdot\|, \widetilde{\|\cdot\|}) \rightsquigarrow (V, \text{dist}, \widetilde{\text{dist}})$

$\Longrightarrow$ $\|\cdot\|$ & $\widetilde{\|\cdot\|}$ ... EQUIVALENT $\Leftrightarrow$ dist, $\widetilde{\text{dist}}$ ... EQUIVALENT

$(\mathbb{K}, |\cdot|)$ ... COMPLETE $\rightsquigarrow$ BANACH SPACE

$\mathbb{K}^d$

$$\|z\|_1 := \sum_{i=1}^{d} |z_i|$$

$$\|z\|_p := \sqrt[p]{\sum_{i=1}^{d} |z_i|^p}, \quad p = 1, 2, \dots$$

$$\|z\|_\infty := \max_{i=1,\dots,d} |z_i|$$

- $\underline{\text{THM}}$ $\dim V < \infty \Longrightarrow \forall$ N.Q ON V ... EQUIVALENT

- $\underline{\text{COR}}$ ALL FINITE-DIMENSIONAL VS ... COMPLETE

SOME FUNCTIONAL SPACES

$$\mathbb{K}^{\mathbb{N}} := \{(z_k)_{k\in\mathbb{N}} \mid \forall k\in\mathbb{N} : z_k \in \mathbb{K}\}$$

$$\ell^{\infty}(\mathbb{K}) := \{z \in \mathbb{K}^{\mathbb{N}} \mid (z_k)_{k\in\mathbb{N}} \ \cdots \ \text{BOUNDED}\}$$

$$c(\mathbb{K}) := \{ \text{---}''\text{---} \mid \text{---}''\text{---} \ \text{CONVERGENT}\}$$

$$c_0(\mathbb{K}) := \{ \text{---}''\text{---} \mid \text{---}''\text{---} \ \text{NULL SEQUENCE}\}$$

$$c_e(\mathbb{K}) := \{ \text{---}''\text{---} \mid \text{---}''\text{---} \ \text{FINITE} \ \text{---}''\}$$

$U \subseteq \mathbb{R}^d$ SUBSET. $\quad \operatorname{supp} f = \overline{\{x \in U \mid f(x) \neq 0\}}$

$$\mathcal{B}(U) := \{f : U \to \mathbb{K} \mid f \ \cdots \ \text{BOUNDED}\}$$

$$\|f\|_\infty := \sup_{x \in U} |f(x)|$$

$1 \leq p < \infty$

$$L^p(U, \mathbb{K}) := \left\{ [f] \in \dots \;\middle|\; \|f\|_p := \sqrt[p]{\int_U \underbrace{|f(x)|}_{x}^{\,p=2} dx} < \infty \right\}$$

$$L^\infty(\text{—}\,''\,) := \left\{ \text{—}\ \cdots\ \text{—}''\text{—}\ \ \|f\|_\infty := \operatorname*{ess\,sup}_{x \in U} |f(x)| < \infty \right\}$$

$$\operatorname{ess\,sup} |f(x)| := \inf \{ M > 0 \mid x \in U : |f(x)| > M \text{ HAS LEBESGUE MEASURE } 0 \}$$

- EXM. $L^2(U, \mathbb{K})$ ... HILBERT SPACE $\qquad L^2$ ... QM
- $\ell^p(\mathbb{K})$, $L^p(U, \mathbb{K})$, $p \neq 2$ ... NOT INNER PRODUCT SPACE
- $C(K, \mathbb{R})$, $\mathbb{K} \neq 0$ $\qquad$ —''— $\quad$ ———
- $c_{cpt}(\mathbb{K})$ ... INNER PRODUCT SPACE BUT NOT HILBERT SPACE

· **DEF** $(\mathcal{H}, \langle \cdot | \cdot \rangle)$

ORTHOGONAL COMPLEMENT OF $V \subset \mathcal{H}$

$$V^{\perp} := \{ u \in \mathcal{H} \mid \forall\, v \in V : \langle u | v \rangle = 0 \}$$

$V \subset \mathcal{H}$ CLOSED $\Longrightarrow$ $\mathcal{H} = V \oplus V^{\perp}$

RECALL,

$(\mathcal{H}_1, \langle \cdot | \cdot \rangle_1)$, $(\mathcal{H}_2, \langle \cdot | \cdot \rangle_2)$ ... THEIR DIRECT SUM IS DEFINED AS THE SET

$$(u_1, u_2) \in \mathcal{H}_1 \times \mathcal{H}_2$$

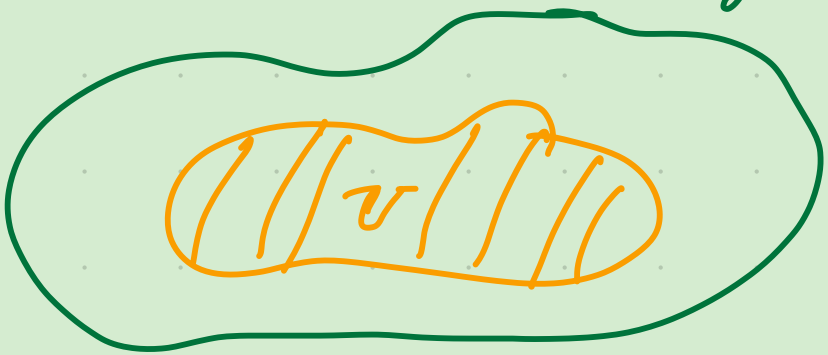
$$\langle (u_1, u_2), (v_1, v_2) \rangle_{\mathcal{H}_1 \times \mathcal{H}_2} := \langle u_1 | v_1 \rangle_1 + \langle u_2 | v_2 \rangle_2$$

$$\bigoplus_{i=1}^{\infty} \mathcal{H}_i := \{ (u_1, u_2, \dots) \mid u_i \in \mathcal{H}_i, \sum \|u_i\|_i^2 < \infty \}$$

• **DEF** $\mathcal{H}$ ... SEPARABLE IF IT ADMITS A COUNTABLE DENSE SEBSET

$U \subset \mathcal{H}$ DENSE $\quad \bar{U}$ $\quad \bar{\mathbb{Q}}$ $\quad \mathbb{R}$

$\|\cdot\|$ $\quad \mathcal{H}$ $\quad |\cdot|$



$C^{\infty}(\mathbb{R}^d)$

• **DEF** $(e_1, e_2, \dots)$ SEQUENCE $\subset \mathcal{H}$
ORTHONORMAL IF $\langle e_i | e_j \rangle = \delta_{ij}$

• **THM** A SEPARABLE $\mathcal{H}$ ADMITS AN ORTHONORMAL BASIS.

OPERATORS

$(\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \leftsquigarrow (\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$, $\quad (\mathcal{K}, \|\cdot\|_{\mathcal{K}}) \leftsquigarrow (\mathcal{K}, \langle\cdot|\cdot\rangle_{\mathcal{K}})$

dom T $\quad\quad\quad\quad$ $(T, \text{dom}, \mathcal{H})$

LINEAR

· <u>DEF</u> $\quad T : \mathcal{H} \longrightarrow \mathcal{K}, \quad u \longmapsto T(u)$

$\mathcal{L}(\mathcal{H}, \mathcal{K})$

$$\|\cdot\| : \mathcal{L}(\mathcal{H}, \mathcal{K}) \to \mathbb{R}, \quad T \mapsto \|T\| := \sup_{u \in \mathcal{H} \setminus \{0\}} \frac{\|Tu\|_{\mathcal{K}}}{\|u\|_{\mathcal{H}}}$$

T ... BOUNDED IF $\|T\| < \infty$

$\mathcal{B}(\mathcal{H}, \mathcal{K})$

T ... UNBOUNDED IF IT IS NOT BOUNDED

• **EXM** $f \in L^\infty(\mathbb{R}^d, \mathbb{K})$

MULTIPLICATION OP BY f

$$M_f : L^p(\mathbb{R}^d, \mathbb{K}) \longrightarrow L^p(\mathbb{R}^d, \mathbb{K}), \qquad u \longmapsto M_f(u) := f \cdot u$$

$$\lambda \cdot u \longmapsto M_f(\lambda u) = f \cdot (\lambda u) = \lambda \cdot f \cdot u \;\checkmark$$

$$\|M_f(u)\|_p = \|f \cdot u\|_p$$

$$\leq \|f\|_\infty \, \|u\|_p \qquad |f| \leq \|f\|_\infty \ldots$$

$$\Rightarrow \|M_f\| \leq \|f\|_\infty \;\checkmark$$

FOR $a < \|f\|_\infty$, SET $A := \{|f| \geq a\}$

$\mathbb{1}_A$ ... CHARACTERISTIC FUNCTION

$$\mathbb{1}_A : A \to \{0,1\}$$

$$x \mapsto \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

$$\|\mathbb{1}_A\|_p = \int_A 1 \, dx > 0$$

$$\|f \mathbb{1}_A\|_p \geq a \|\mathbb{1}_A\|_p$$

$$\hookrightarrow \forall a < \|f\|_\infty, \quad \|M_f\| \geq a$$

$$\boxed{\therefore \|M_f\| = \|f\|_\infty}$$

• <u>DEF</u> $T \in \mathcal{B}(\mathcal{H}, \tilde{\mathcal{H}})$, $S \in \mathcal{B}(\tilde{\mathcal{H}}, \mathcal{K})$

$\circ : \mathcal{B}(\mathcal{H}, \tilde{\mathcal{H}}) \times \mathcal{B}(\tilde{\mathcal{H}}, \mathcal{K}) \longrightarrow \mathcal{B}(\mathcal{H}, \mathcal{K})$, $(T, S) \longmapsto S \circ T$

$$\|S \circ T\| \leq \|S\| \, \|T\|$$

• **REM** $(\mathcal{B}(\mathcal{H}), \circ)$ … ALGEBRA
$\lambda T, \quad S+T, \quad S\cdot T \in \mathcal{B}(\mathcal{H})$
$\Longrightarrow$ Alg QM (AQM)
AQFT [HAAG]

$V \subset \mathcal{H}$ DENSE
$\overline{V} = \mathcal{H}$
$\|h - u\| < \varepsilon$

• **EXM** $f : \mathbb{R}^d \longrightarrow \mathbb{K}$ "MEASURABLE"

$$\operatorname{dom}(M_f) := \{ u \in L^2(\underbrace{\mathbb{R}^d}_{K}, \mathbb{K}) \mid f \cdot u \in L^2(\underbrace{\mathbb{R}^d}_{K}, \mathbb{K}) \}$$

$= |\xi|^2$

$$M_f : \operatorname{dom}(M_f) \longrightarrow L^2(\mathbb{R}^d, \mathbb{K}), \quad u \longmapsto M_f(u) := f \cdot u$$

$$M_f \text{ … BOUNDED } \Longleftrightarrow f \in L^\infty(\mathbb{R}^d, \mathbb{K})$$

$\not\Leftarrow$

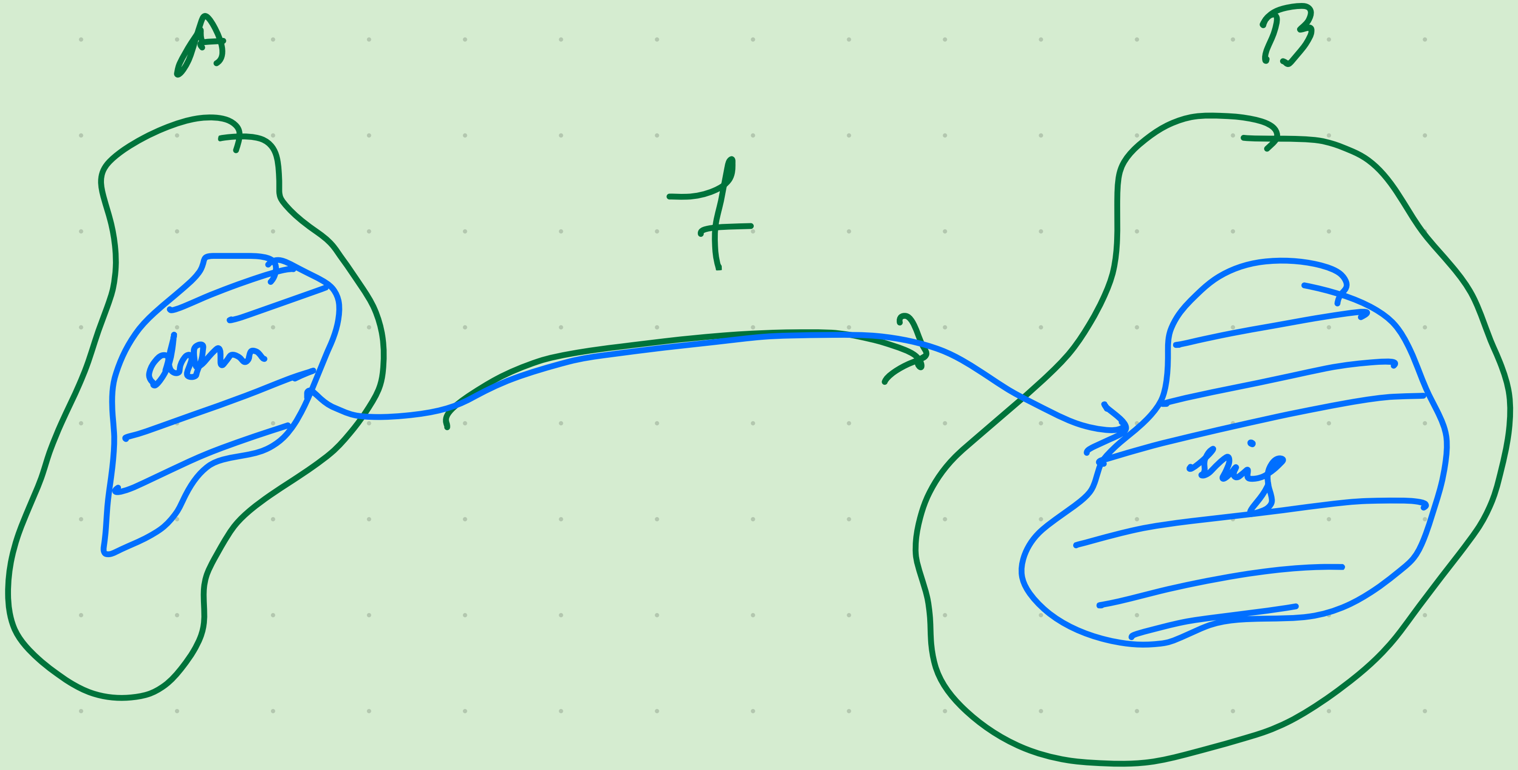
$$f \stackrel{!}{=} 1 \Longrightarrow \|f\|_\infty = \infty$$

$K \subset \mathbb{R}^d$ CPT $\Longrightarrow \|f\|_\infty < \infty$ $= L^2(K, \mathbb{K})$

$\Longrightarrow M_f : \operatorname{dom} M_f \longrightarrow L^2(K, \mathbb{K})$ bounded

COMPACT SPACE $\longleftrightarrow$ SEQUENCE HAS CONVERGENT SUB-SEQ

COMPLETE $\longleftrightarrow$ $\exists$ SEQ THAT CONVERGES ... VECTOR



$$(u_n)_n \longrightarrow u$$

$$T u_n \overset{?}{\longrightarrow} u$$

DEF $f \in L^1(\mathbb{R}^d, \mathbb{K})$

$$\hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx$$

$$\mathcal{F} : L^1(\mathbb{R}^d, \mathbb{K}) \longrightarrow ?$$

SCHWARTZ SPACE $\mathcal{S}(\mathbb{R}^d) := \{ f \in C^\infty(\mathbb{R}^d, \mathbb{K}) \mid \forall \alpha, \beta \in \mathbb{N}_0^d : \|x^\alpha D^\beta f\|_\infty < \infty \}$

$\alpha = (\alpha_1, \dots, \alpha_d)$,

$|\alpha| := |\alpha_1| + \dots |\alpha_d|$, $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$

MULTI-INDEX NOTATION

$$D^\beta := (-i)^{|\beta|} \frac{\partial^{|\beta|}}{\partial(x^1)^{\beta_1} \cdots \partial(x^d)^{\beta_d}}$$

$$u \in \mathcal{S}(\mathbb{R}^d, \mathbb{K}) \implies \widehat{(D_x^\alpha u)}(\xi) = (i\xi)^\alpha (\mathcal{F}u)(\xi)$$

$$\widehat{(x^\alpha u)}(\xi) = (iD_\xi)^\alpha (\mathcal{F}u)(\xi)$$

$$\implies \hat{\cdot} : \mathcal{S}(\mathbb{R}^d, \mathbb{K}) \longrightarrow \mathcal{S}(\mathbb{R}^d, \mathbb{K})$$

$$\overline{\mathcal{S}}(\mathbb{R}^d, \mathbb{K}) = L^2(\mathbb{R}^d, \mathbb{K})$$

- <u>DEF</u> $(\mathcal{H}, \|\cdot\|)$, $(\mathcal{K}, \|\cdot\|)$

$$T \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \text{ ISOMETRY } \iff \|Tu\|_{\mathcal{K}} = \|u\|_{\mathcal{H}}$$

$$\|T\| = 1$$

$$T \ldots \text{ BIJECTIVE IF } \exists\ T^{-1} \underset{\notin}{\in} \mathcal{B}(\mathcal{K}, \mathcal{H})$$

$$(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}}), \ (\mathcal{K}, \langle \cdot | \cdot \rangle_{\mathcal{K}})$$

$$T \ldots \text{ UNITARY } := \text{ BIJECTIVE \& ISOMETRY}$$

$$\langle Tu | Tv \rangle = \langle u | v \rangle = \langle T^{*}T u | v \rangle$$

$$\|T\| = 1$$

- <u>THM [PLANCHEREL]</u> $\mathcal{F} : L^2(\mathbb{R}^d, \mathbb{K}) \longrightarrow L^2(\mathbb{R}^d, \mathbb{K})$ UNITARY

$$\Delta = -\frac{\partial^2}{\partial x^2}$$

$$|\psi\rangle$$

$$\Longrightarrow \mathcal{F}(\Delta u) = |\xi|^2 \mathcal{F}(u) = M_{|\xi|^2} \mathcal{F}(u)$$

$$\mathrm{dom}(\Delta) := \{ u \in L^2(\mathbb{R}^d, \mathbb{K}) \mid |\xi|^2 \mathcal{F}(u) \in L^2(\mathbb{R}^d, \mathbb{K}) \}$$

$$\Delta : \mathrm{dom}(\Delta) \longrightarrow L^2(\mathbb{R}^d, \mathbb{K}), \quad u \longmapsto \Delta(u) := M_{|\xi|^2} \hat{u}$$

$\| M_{|\xi|^2} \| = \infty \Longrightarrow \Delta$ ... UNBOUNDED OP

$$\hat{p} |p\rangle = p |p\rangle$$

QM

Math rep universe

• DEF IMAGE / RANGE

$$\operatorname{img}(T) := T(\operatorname{dom} T) = \{T(u) \mid u \in \operatorname{dom} T\}$$

KERNEL / NULL SPACE

$$\ker(T) := \{u \in \operatorname{dom} T \mid T(u) = 0\}$$

$\hookrightarrow$ $\ker T,\ \operatorname{img} T \subset \mathcal{H}$

• DEF RESTRICTION ... $T : \operatorname{dom} T \to \mathcal{H}$ ON $\mathcal{D} \subset \operatorname{dom} T$

$$T_{\restriction \mathcal{D}} : \mathcal{D} \to \mathcal{H}, \quad u \mapsto T_{\restriction \mathcal{D}}(u) := T(u)$$

$S : \operatorname{dom} S \to \mathcal{H}$

$$S = T \iff \operatorname{dom} S = \operatorname{dom} T, \quad \forall u \in \operatorname{dom} S : \quad S(u) = T(u)$$

T ... EXTENSION OF $S \iff S = T_{\restriction \operatorname{dom} S}$

$S \subseteq T$ WHEN $\operatorname{dom} S \subseteq \operatorname{dom} T, \ \forall u \in \operatorname{dom} S : \ S(u) = T(u)$

"Op" $\not\Longleftrightarrow$ "MATRICES"

$$\mathcal{B}(\mathcal{H}) \ni T = \begin{pmatrix} T^1_1 & & & \\ & T^i_j & & \\ & & \ddots & \\ & & & \ddots \end{pmatrix} \overset{?}{\longleftrightarrow} M = \begin{pmatrix} M^1_1 & \cdots & \cdots & M^1_m \\ \ddots & M^i_j & & \\ & & & \\ M & & \ddots & M^n_m \end{pmatrix}$$

$\times \longleftarrow \dim(\mathrm{dom}) = \dim(\mathrm{img}) + \dim(\ker)$

POSITION OP ... QM

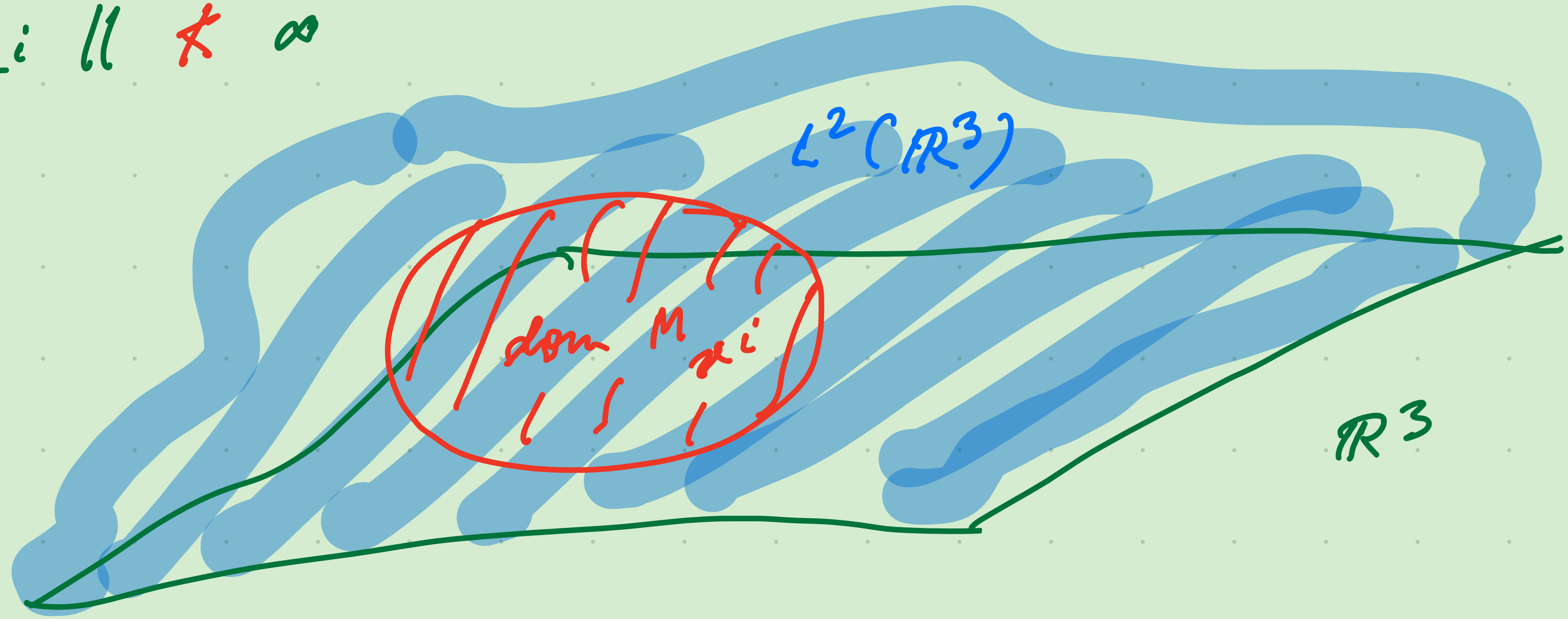
- DEF $\mathcal{H} := L^2(\mathbb{R}^d)$, $\quad i = 1, 2, \ldots, \underset{3}{d}$, $\quad x = (x^1, \ldots, x^d) \in \mathbb{R}^d$

$$\mathrm{dom}(M_{x^i}) := \{ u \in \mathcal{H} \mid x^i \cdot u \in \mathcal{H} \}$$

$$M_{x^i} : \mathrm{dom}(M_{x^i}) \to \mathcal{H}, \quad u \mapsto M_{x^i}(u) := x^i \cdot u$$

POSITION OP

$\implies \|M_{x^i}\| \not< \infty$



$C^\infty(U, \mathbb{K})$, $U \subset \mathbb{R}^d$ OPEN

$\mathbb{R} : |\cdot|$

$\mathcal{H} : \|\cdot\|$

$|||\cdot|||_{\alpha, K} : C^\infty(U, \mathbb{K}) \longrightarrow \mathbb{R}$,

$$u \longmapsto |||u|||_{\alpha, K} := \sup_{x \in K \subset_{cpt} U} |(D^\alpha u)(x)|$$

$(u_n) \longrightarrow u$ ... $C^\infty(U, \mathbb{K}) \iff \forall$ DERIVATIVES CONVERGES UNIFORMLY

$\mathcal{E}(U,\mathbb{K}) := (C^\infty(U,\mathbb{K}), |||\cdot|||)$ FRÉCHET SPACE

$$\operatorname{supp} u := \overline{\{x \in U \mid u(x) \neq 0\}}$$

$$C_c^\infty(U,\mathbb{K}) := \{u \in C^\infty(U,\mathbb{K}) \mid \operatorname{supp} u \text{ CPT}\}$$

$K_1 \subset K_2 \subset K_3 \subset \cdots$ CPT EXHAUSTION $\bigcup_i K_i = U$

$$C_c^\infty(K_i) = \{ \text{—} u \text{—} \mid \operatorname{supp} u \subset K_i\}$$

FOR EACH $u \in C_c^\infty(U)$ $\exists\, k \mid \forall\, i \geq k$, $\operatorname{supp} u \subset K_i$

$$C_c^\infty(K_1) \hookrightarrow C_c^\infty(K_2) \hookrightarrow \cdots$$

CONSIDER REL TOP $C_c^\infty(K_i) \subset \mathcal{E}(U)$

$(u_n)_n \longrightarrow u \; \dots \; C_c^\infty(U, \mathbb{K})$ IF

$\exists$ K CPT | $\operatorname{supp} u_n \subset K_n$ & $\forall$ DERIVATIVES OF u_n

$\operatorname{supp} u \subset K$ CONVERGES UNIFORMLY IN K

$\mathcal{D}(U, \mathbb{K}) := \left(C_c^\infty(U, \mathbb{K}), \text{ INDUCTIVE LIMIT TOPOLOGY} \right)$

DISTRIBUTION

- <u>DEF</u> DISTRIBUTION := CONTINUOUS (LIN) FUNCTIONAL ON $C_c^\infty(U, \mathbb{K})$

C^∞

$$u : C_c^\infty(U, \mathbb{K}) \longrightarrow \mathbb{C}, \quad \phi \longmapsto u(\phi)$$

$$|u(\phi)| \leq \text{cst} \sum_{|\alpha| \leq k} \sup_{x \in K} |(D^\alpha \phi)(x)|$$

$C_c^{\infty\prime}(U, \mathbb{K}) :=$ TOPOLOGICAL DUAL OF $C_c^\infty(U, \mathbb{K})$

$C^{\infty\prime}$ $\qquad$ C^∞

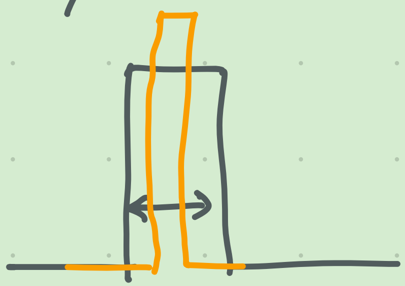
$\mathcal{D}'(U, \mathbb{K}) := \left(C_c^{\infty\prime}(U, \mathbb{K}),\ \text{TOPOLOGY} \right)$

$\mathcal{E}'$ $\qquad$ $C^{\infty\prime}$

· EXM: $\delta_a : C_c^{\infty}(U, \mathbb{K}) \longrightarrow \mathbb{K}, \quad \phi \longmapsto \delta_a(\phi) := \phi(a)$

$\Longrightarrow |\delta(\phi)| = |\phi(0)|$

$$\int_U \delta(x-a)\, \phi(x)\, dx \cancel{\overset{?}{=}} \phi(a), \qquad \int_U \delta(x-a)\, dx = 1$$



$$\int_U \delta(x-a)\, \phi(x)\, dx = \delta_a(\phi)$$

$f \cdot u(\phi) := u(f \cdot \phi)$

$f \cdot \delta(\phi) = \delta(f \cdot \phi) = f(0)\, \phi(0) = f(0)\, \delta(\phi)$

$u \in C^\infty(U, \mathbb{K})$, $\quad \phi \in C_c^\infty(U, \mathbb{K}) \implies u \cdot \phi \in C_c^\infty(U, \mathbb{K})$

$u : C_c^\infty(U, \mathbb{K}) \to \mathbb{K}, \quad \phi \mapsto u(\phi) := \int_U u(x)\, \phi(x)\, dx$

$\hookrightarrow u \in C_c^{\infty'}(U, \mathbb{K})$

$\boxed{\text{REGULAR}}$ DISTRIBUTION $\qquad C^\infty(U, \mathbb{K}) \hookrightarrow C_c^{\infty'}(U, \mathbb{K})$

$$\delta(\phi) \cancel{=} \int_U \underbrace{\delta(x)}_{= \delta(x-0)} \phi(x)\, dx \cancel{=} \phi(0)$$

HYDROGEN ATOM $\qquad \dfrac{1}{x^n} \qquad x = 0$ $\longrightarrow$

d

$$\int \frac{\partial u}{\partial x^i} \phi \, dx = - \int u \frac{\partial \phi}{\partial x^i} dx$$

WEAK DERIVATIVE

• <u>DEF</u> $\partial_i : C^\infty(V, \mathbb{K}) \circlearrowleft$

$\boxed{\partial_i'} : C_c^{\infty\prime}(V, \mathbb{K}) \circlearrowleft, \quad u \longmapsto \partial_i' u(\phi) := -u(\partial_i \phi)$

DISTRIBUTIONAL DERIVATIVE

$L : C^\infty(V, \mathbb{K}) \circlearrowleft, \quad \phi \longmapsto L(\phi)$

$L' : C^{\infty\prime}(V, \mathbb{K}) \circlearrowleft, \quad u \longmapsto L' u(\phi) := u(L\phi)$

DIFFERENTIAL OP ... DISTRIBUTIONAL SENCE

SELF-ADJOINT OP

$T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$

- **DEF** GRAPH OF T

$$\mathcal{G}(T) := \{(u, Tu) \mid u \in \operatorname{dom} T\} \underset{\text{space}}{\overset{\text{lin}}{\subset}} \mathcal{H} \oplus \mathcal{K}$$

$\forall$ INFO ... T

$$\langle u | v \rangle_T := \langle u | v \rangle_{\mathcal{H}} + \langle Tu | Tv \rangle_{\mathcal{K}}, \quad u, v \in \operatorname{dom} T$$

GRAPH NORM

$$\|u\|_T := \sqrt{\|u\|_{\mathcal{H}}^2 + \|Tu\|_{\mathcal{K}}^2}$$

- **DEF** $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ ... CLOSED IF $\mathcal{G}(T) \underset{\text{subspace}}{\overset{\text{closed}}{\subset}} \mathcal{H} \oplus \mathcal{K}$

T ... CLOSABLE IF $\exists$ CLOSED $S \in \mathcal{L}(\mathcal{H}, \mathcal{K}) \mid T \subseteq S$

• <u>DEF</u> T ... CLOSABLE $\mathcal{L}(\mathcal{H}, \mathcal{K})$

$\bar{T}$ ... CLOSED $\mathcal{L}(\mathcal{H}, \mathcal{K})$! $\qquad \overline{\mathcal{G}(T)} = \mathcal{G}(\bar{T})$

CLOSURE OF T

$$T : \mathbb{C}^d \to \mathbb{C}^d \qquad T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1d} \\ T_{21} & T_{22} & \cdots & T_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ T_{d1} & T_{d2} & \cdots & T_{dd} \end{pmatrix}$$

$\{\text{BASIS}\} \to T\{\dots\} \longleftrightarrow T =$

$\dagger \qquad \bar{*}$

$$T^* = \underbrace{T_{ij}^{cc} \quad \text{rows} \leftrightarrow \text{columns}}_{?\, T \in \mathcal{L}(\mathcal{H})}$$

$\langle u | v \rangle \in \mathbb{C}$

$\langle u | u \rangle \in \mathbb{R}$

$\mathbb{C} \ni z \mapsto z^{cc} = (x + iy)^{cc} := x - iy$

$\mathbb{C}$-lin $\qquad =: u^{cc}$

$\mathcal{H} \ni u \longmapsto$ dual space $\mathcal{H}' \ni \phi : \phi(u) \in \mathbb{C}$

$\neq \mathcal{H}^{cc} \qquad z^{cc} z \in \mathbb{R}$

· **DEF** $T \in \mathcal{L}(\mathcal{H}, \mathcal{K})$.

bounded

$= \langle w | T^* v \rangle$

$$\mathrm{dom}\,(T^*) := \{ v \in \mathcal{K} \mid \exists\ u \in \mathcal{H} \mid \forall\ w \in \mathrm{dom}\,T : \langle Tw | v \rangle_{\mathcal{K}} = \langle w | u \rangle_{\mathcal{H}} \}$$

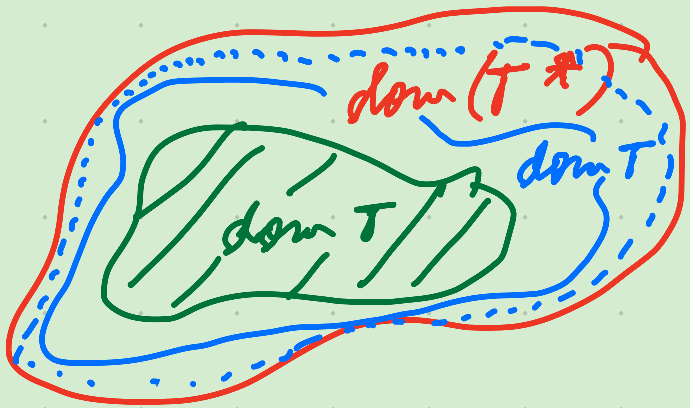
$T^* v := u \implies T^* \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ ... ADJOINT OF T

$T \in \mathcal{L}(\mathcal{H})$ * $\not\equiv$: SA ... (PHYSICS)"

T ... SYMMETRIC IF $T \subseteq T^*$

SELF-ADJOINT $T = T^*$

ESSENTIALLY — " — $\bar{T} = T^*$



$T \in \mathcal{B}(\mathcal{H}, \mathcal{H}) \Longrightarrow \operatorname{dom} T = \mathcal{H}$

$\in$ MATRIX

- **EXM** $f : \mathbb{R} \xrightarrow{C^{\mathbb{R}}} \mathbb{C}, \quad M_f$

$$(M_f)^* = M_{\bar{f}} \quad \text{cc}$$

ANTI-LIN

$$\langle M_f w | v \rangle = \langle f w | v \rangle = \langle w | \bar{f} v \rangle = \langle w | M_{\bar{f}} v \rangle$$

$f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ ... ABSOLUTELY CONTINUOUS $\Longleftrightarrow$

$\exists f \in L^1(a, b) \mid$

$$\forall x \in [a, b] : \quad f(x) = f(a) + \int_a^x g(t)\, dt$$

$AC[a, b] := \{ \text{—} n \text{—} \}$

IF $f \in AC[a,b]$ THEN $f \in C([a,b],\mathbb{R})$ & f ... A.E. DIFF.

$f'(x) = g(x)$ AE ... $[a,b]$

DERIVATIVE OF f

$$H^1(a,b) := \{ f \in AC[a,b] \mid f' \in L^2(a,b) \}$$

$$H^2(a,b) := \{ f \in C^1([a,b],\mathbb{R}) \mid f' \in H^1(a,b) \}$$

INTEGRATION BY PARTS

$\forall f,g \in H^1(a,b)$:

$$\boxed{\begin{aligned} &\langle f' \mid g \rangle_{L^2} + \langle f \mid g' \rangle_{L^2} \\ &= \bar{f} g \Big|_a^b \\ &= \bar{f}(b)\, g(b) - \bar{f}(a)\, g(a) \end{aligned}}$$

$$\forall\ f, g \in H^2(a,b): \boxed{\langle f'' \mid g\rangle_{L^2} - \langle f \mid g''\rangle_{L^2}}$$

$$= \left(\overline{f'}\, g - \overline{f}\, g'\right)\Big|_a^b$$

$$\boxed{= \overline{f'(b)}\ g(b) - \overline{f(b)}\ g'(b) - \overline{f'(a)}\ g(a) + \overline{f(a)}\ \overline{g'(a)}}$$

· <u>EXM</u> $T : H^1_0(a,b) \longrightarrow L^2(a,b), \quad u \longmapsto T(u) := -i\, u'$

$$H^1_0(a,b) = \{ u \in H^1(a,b) \mid u(a) = 0 = u(b) \}$$

$T^2 : H^2_0(a,b) \longrightarrow L^2(a,b), \quad u \longmapsto T^2(u) := -u''$

$$H^2_0(a,b) = \{ u \in H^2(a,b) \mid u(a) = 0 = u(b),\ \ u'(a) = 0 = u'(b) \}$$

· Lem $\boxed{\begin{aligned} \operatorname{img}(T)^{\perp} &\subseteq \mathbb{C}\cdot 1, \\ \operatorname{img}(T^2)^{\perp} &\subseteq \mathbb{C}\cdot 1 + \mathbb{C}\cdot x \end{aligned}}$

$f_1(x) := 1$

$f_2(x) := x$ for $x \in [a,b]$

Pf IT IS SUFFICIENT TO PROVE THAT $(\mathbb{C}\cdot 1)^{\perp} \subseteq \operatorname{img}(T)$ & $(\mathbb{C}\cdot 1 + \mathbb{C}\cdot x)^{\perp} \subseteq \operatorname{img}(T^2)$.

SUPPOSE $h_1 \in (\mathbb{C}\cdot 1)^{\perp}$, $h_2 \in (\mathbb{C}\cdot 1 + \mathbb{C}\cdot x)^{\perp}$. DEFINE FUNCTIONS ON $[a,b]$:

$$g_1(x) := \int_a^x h_1(t)\,dt, \qquad g_2(x) := \int_a^x \left(\int_a^t h_2(s)\,ds \right) dt$$

OBSERVE THAT $h_1, h_2 \in L^2(a,b)$

$$\Longrightarrow g_1 \in H^1(a,b),\ g_1' = h_1$$

$$g_2 \in H^2(a,b),\ g_2'' = h_2$$

CHECK ... $g_1(a) = 0 = g_2(a)$. $g_1(b) = \langle h_1, 1 \rangle = 0$,

$$g_2'(b) = \langle h_2 | 1 \rangle = 0$$

$$\Longrightarrow \quad g_2'(x)\, x \Big|_a^b = 0 \quad \cdots\cdots \text{ INTEGRATION BY PARTS}$$

$$g_2(b) = \int_a^b g_2'(t)\, dt$$

$$= \langle g_2' | 1 \rangle_{L^2}$$

$$= -\langle g_2'' | x \rangle_{L^2}$$

$$= -\langle h_2 | x \rangle_{L^2}$$

$$= 0$$

$$\Longrightarrow \quad g_1 \in \operatorname{dom} T, \quad g_2 \in \operatorname{dom}(T^2)$$

$$\Longrightarrow \quad T(i g_1) = h_1 \in \operatorname{img} T, \quad T^2(-g_2) = h_2 \in \operatorname{img}(T^2) \quad \blacksquare$$

• Lem $\boxed{\begin{array}{l} \mathrm{dom}(T^*) = H^1(a,b), \quad T^* v := -i v' \\ \mathrm{dom}((T^2)^*) = \mathrm{dom}((T^*)^2) = H^2(a,b), \quad (T^2)^* v := -v'' \end{array}}$

Pf LET $u \in \mathrm{dom}\, T$. SINCE $u(a) = 0 = u(b)$. ... INTEGRATION BY PARTS ... ↯

$$\langle Tu | v \rangle = -i \langle u' | v \rangle = i \langle u | v' \rangle = \langle u | -iv' \rangle$$

DEF ... $T^* \Longrightarrow v \in \mathrm{dom}\, T^*$ & $T^*(v) = -iv'$.

LET $v \in H^2(a,b) \xrightarrow{\text{DEF}} v, v' \in H^1(a,b)$. ... USE .... TO v AND THEN TO $T^* v = -iv'$, WE CONCLUDE THAT $v \in \mathrm{dom}(T^{*2})$ AND $T^{*2} v = -v''$.

LET $u_1 \in \mathrm{dom}\, T$, $u_2 \in \mathrm{dom}\, T^2$. $u_1(a), u_1(b), u_2(a), u_2(b)$, $u_2'(a)$, $u_2'(b)$ ... VANISHES, SO ... "IBP"

$$-\langle u_1' \mid g_1 \rangle = \langle u_1 \mid g_1' \rangle = \langle u_1 \mid h_1 \rangle = \langle u_1 \mid T^* v_1 \rangle = \langle T u_1 \mid v_1 \rangle$$
$$= \langle -i\, u_1' \mid v_1 \rangle$$

$$\langle u_2'' \mid g_2 \rangle = \langle u_2 \mid g_2'' \rangle = \langle u_2 \mid (T^2)^* v_2 \rangle = \langle T^2 u_2 \mid v_2 \rangle = \langle -u_2'' \mid v_2 \rangle$$

$$\because \langle -i\, u_1' \mid v_1 - i g_1 \rangle = 0 = \langle u_2'' \mid v_2 + g_2 \rangle$$
$$\Longrightarrow v_1 - i g_1 \in \operatorname{img}(T)^\perp \underset{\text{Lem}}{\subseteq} \mathbb{C} \cdot 1$$
$$v_2 + g_2 \in \operatorname{img}(T^2)^\perp \subseteq \mathbb{C} \cdot 1 + \mathbb{C} \cdot x$$

$$\because g_1, 1 \in H^1(a,b), \quad g_2, 1, x \in H^2(a,b)$$
$$\Longrightarrow g_1 \in H^1(a,b), \quad g_2 \in H^2(a,b)$$

$\ldots \ T^*$,

$$H^2(a,b) \subseteq \operatorname{dom}((T^*)^2), \quad T^{*2} v = -v''$$
$$\operatorname{dom}((T^2)^*) \subseteq H^2(a,b)$$
$$\because (T^*)^2 \subseteq (T^2)^* \Longrightarrow \ldots (T^2)^* \ \& \ (T^*)^2 \quad \blacksquare$$

• LEM $\boxed{T^{**} = T}$ $\Longleftarrow$ T … CLOSED

PF … FACT $T \subseteq T^{**}$ … SHOW THAT $T^{**} \subseteq T$. LET $w \in \operatorname{dom}(T^{**})$ …

$\underset{\text{Lem}}{T \subseteq T^*} \Rightarrow T^{**} \subseteq T^*$, $w \in \operatorname{dom}(T^*) = H^1(a,b)$,

$(T^*)^* w = \underset{\text{Lem}}{-i w'}$

$\forall\, v \in \operatorname{dom} T^*$:

$0 = \langle T^* v \mid w \rangle - \langle v \mid T^{**} w \rangle$

$= \langle -i v' \mid w \rangle - \langle v \mid -i w' \rangle \overset{\text{IBP}}{=} -i\left(\overline{v(b)}\, w(b) - \overline{v(a)}\, w(a)\right)$

$\because v(b), v(a) \in \operatorname{dom}(T^*)$ … ARBITRARY $\Rightarrow w(a) = 0 = w(b)$

$\therefore w \in \operatorname{dom} T$

$\blacksquare$

• **DEF** $(T, \mathrm{dom}(T) \subset \mathcal{H})$ SYMMETRIC

$\mathrm{dom}(T)$ ... CALLED THE CORE OF T IF T ... ESA

$$T_0 := T\restriction_{\mathcal{D}_0}, \quad \mathcal{D}_0 := \{u \in C^\infty([a,b], \mathbb{R}) \mid u(a) = 0 = u(b)\}$$

• **LEM** $\boxed{\mathcal{D}_0 \text{ ... CORE FOR } T, \text{ SO } \overline{T_0} = T}$

Pf LET $u \in \mathrm{dom}$;

$\because u' \in L^2(a,b)$

$\implies \exists\ (v_n)_{n \in \mathbb{N}}$ ... $C^\infty([a,b], \mathbb{R})$ | $v_n \to u'$ IN $L^2(a,b)$

$$u_n(x) := \int_a^x v_n(t)\, dt - (x-a)(b-a)^{-1} \int_a^b v_n(t)\, dt$$

HÖLDER INEQUALITY $\implies v_n \to u' \in L^1(a,b)$

$$\Longrightarrow \int_a^b v_n(t)\,dt \longrightarrow \int_a^b u'(t)\,dt = u(b) - u(a) = 0$$

$$\therefore\ T(u_n) = -i v_n + i(b-a)^{-1} \int_a^b v_n(t)\,dt \longrightarrow iu' = Tu \ \dots\ L^2(a,b)$$

$\because\ u(a) = 0$

$$x \in [a,b] : \ |u_n(x) - u(x)| \leq \int_a^b |v_n(t) - u'(t)|\,dt + \int_a^b \left| v_n(t)\,dt \right|$$

$\because\ v_n \longrightarrow u' \ \dots\ L^1(a,b) \Longrightarrow \text{RHS} \longrightarrow 0$ AS $n \longrightarrow \infty$

$\therefore\ u_n(x) \longrightarrow u(x)$ UNIFORMLY ... $[a,b]$

$\Longrightarrow u_n \longrightarrow u \ \dots\ L^2(a,b)$

$\Longrightarrow T_0 u_n \longrightarrow T u_n \longrightarrow T u \ \dots\ L^2(a,b)$ $\boxtimes$

- **EXM** ... M_f

  M_f ... SA $\Leftrightarrow$ $f : \mathbb{C} \to \mathbb{R}$ A.E.

- **EXM** Δ ... UNITARY EQUIVALENT $M_{|\xi|^2} \Rightarrow$ SA

  $$\operatorname{dom}(M_{|\xi|^2}) = \{ u \in L^2(\mathbb{R}^d) \mid |\xi|^2 \hat{u} \in L^2(\mathbb{R}^d) \} = H^2(\mathbb{R}^d)$$

$\Rightarrow (\Delta, H^2(\mathbb{R}^d))$ ... SA

- **LEM** FOR ... ESAO, THE CLOSURE IS THE UNIQUE SA EXTENSION

- **EXM** $\Delta : C_c^\infty(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d)$

  $$\operatorname{dom}(\Delta^*) := \{ u \in L^2(\mathbb{R}^d) \mid \phi \in C_c^\infty(\mathbb{R}^d) : \underbrace{\langle u \mid \Delta\phi \rangle}_{L^2(\mathbb{R}^d, dx)} \text{ ... BOUNDED} \}$$

  PLANCHEREL THM ''

  $$\langle \hat{u} \mid |\xi|^2 \hat{\phi} \rangle_{L^2(\mathbb{R}^d, \frac{d\xi}{2\pi^d})}$$

$\Longrightarrow \quad u \in \operatorname{dom}(\Delta^*) \iff u \in L^2(\mathbb{R}^d) \mid v - |\xi|^2 \hat{u} \ldots \perp \mathcal{F}(C_c^\infty(\mathbb{R}^d))$

$\Longrightarrow \quad v = |\xi|^2 \hat{u}$

$\Uparrow$

$\underset{\text{dense}}{C_c^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)}, \quad \mathcal{F} \ldots$ UNITARY

$u \in \operatorname{dom}(\Delta^*) \iff |\xi|^2 \hat{u} \in L^2(\mathbb{R}^d)$

$\operatorname{dom}(\Delta^*) = H^2(\mathbb{R}^d)$

$\because (\Delta, H^2(\mathbb{R}^d)) \ldots SA \Longrightarrow (\Delta, \underbrace{C_c^\infty(\mathbb{R}^d)}_{\text{CORE}}) \ldots ESA$

• <u>LEM</u> $T, S \in SA(\mathcal{H})$

$S \in \mathcal{B}(\mathcal{H})$

$\Longrightarrow T + S \in SA(\mathcal{H}), \quad \operatorname{dom}(T+S) = \operatorname{dom}(T)$

$T + S \in ESA(\mathcal{H})$ ON A CORE FOR T

• <u>EXM</u> $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$

M_V

$\Longrightarrow M_V \in \mathcal{B}_{SA}(L^2(\mathbb{R}^d, \mathbb{R}))$

$\overset{\text{Lem}}{\Longrightarrow}$ $\Delta + V$ "=" $\Delta + M_V$ ... SA ON $H^2(\mathbb{R}^d)$

ESA —"— $C_c^\infty(\mathbb{R}^d, \mathbb{R})$

CRITERIA FOR SA

• <u>THM</u> $T \in$ SYMM

$z \in \mathbb{C}$ STRICTLY $\mathbb{C}$

TFAE: (I) T SA;

(II) T ... CLOSED & BOTH $T^* - z$ & $T^* - \bar{z}$ ... INJECTIVE,

(III) BOTH $T - z$ & $T - \bar{z}$ ... BOUNDED INVERSES;

(IV) BOTH $T - z$ & $T - \bar{z}$ ... SURJECTIVE.

• **THM** $T \in$ SYMM

$z \in \mathbb{C}$ STRICTLY $\mathbb{C}$

TFAE:

(I) T ESA;

(II) BOTH $T^* - z$ & $T^* - \bar{z}$ ... INJECTIVE;

(III) BOTH $T - z$ & $T^* - \bar{z}$ ... DENSE RANGE.

• **EXM** $\Delta : C_c^\infty(0,1) \longrightarrow L^2(0,1)$

CHECK: Δ ... SYMM : IBP

... FROM EARLIER EXMs: IT IS SUFFICIENT TO PRODUCE A NON-ZERO ELEMENT IN $L^2(0,1)$ THAT IS ORTHOGONAL TO $\operatorname{img}(\Delta - z)$ FOR z STRICTLY $\mathbb{C}$.

LET $u \in C^\infty[0,1]$, $\phi \in C_c^\infty(0,1)$:

$$\langle u \mid (\Delta - z)\phi\rangle = \langle u'' - \bar{z} u \mid \phi\rangle = - \langle u'' + \bar{z} u \mid \phi\rangle$$

$u'' + \bar{z} u = 0$ ... ADMITS NON-TRIV SOLS, E.G.,

$$u(x) = e^{i\sqrt{\bar{z}}\, x}$$

SOLS $\perp$ ring $(\Delta - \bar{z})$

$\xRightarrow{\text{Thm}}$ $\Delta \notin$ ESA $\blacksquare$

SPECTRUM

LIN ALG: $V = \mathbb{C}^d$, $\quad L : V \to V$, $\quad L v = \lambda v$ EV

- <u>DEF</u> $L \in \mathcal{L}(\mathcal{H})$ "CLOSED LIN OP"

  - $\lambda \in \mathbb{C}$ ... BELONGS TO RESOLVEN SET $\rho(L)$ OF L IF $L - \lambda \mathbb{1}$ HAS ... BOUNDED INVERSE $(L-\lambda)^{-1}$ DEFINED EVERYWHERE ON $\mathcal{H}$, ... CALLED THE RESOLVENT OF L AT λ AND DENOTED BY $R_\lambda(L)$

  - $\sigma(L) := \mathbb{C} \setminus \rho(L)$ ... SPECTRUM OF L.

  - $\sigma_p(L) := \{\lambda \in \mathbb{C} \mid \ker(L-\lambda) \neq \{0\}\}$ ... POINT SPECTRUM OF L

WE CALL $\lambda \in \sigma_p(L)$ AN EIGENVALUE, $\dim (L-\lambda) =:$ MULTIPLICITY, AND ANY NON-ZERO ELEMENT IN EIGENVECTOR OF L AT λ.